ON THE ANALYTIC CONSTRUCTION OF AN OPTIMAL CONTROL IN A SYSTEM WITH TIME LAGS

(OB ANALITICHESKOM KONSTRUIROVANII OPTIMAL'NOGO REGULIATORA V SISTEME S ZAPAZDYVANIIAMI VREMENI)

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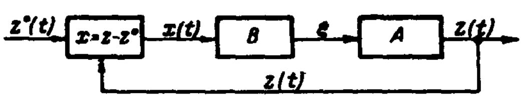
The author considers the problem of the construction of guidance effect in a system that controls an object whose motion is described by linear differential equations with time lag. The optimal control is achieved under conditions of asymptotic stability of the given motion and for the minimum time integral of the square error in the coordinates of the controlled quantity and the guiding action. The solution is based on Liapunov's [1,2] method developed for equations with time lag [3] and modernized in accord with the principles of dynamic programming [4]. It is shown that the optimal reaction of the control must take place at every instant of time t in the form of a functional of functions describing the behavior of the control object during the preceding time interval $t - h \leqslant \tau \leqslant t$ (h is the lag). An explicit expression of this functional is given. The existence of a solution is established by the method of the deformation of the system [5], and an approximate method is given for the construction of the optimum control. The results generalize the investigations of Letov [6] to systems with time lag.



1. Preliminary remarks. Let us consider a control system (see figure) in which the disturbed motion is described by the differential equations

$$\frac{dx_i}{dt} = \sum_{j=1}^{n} a_{ij}x_j(t) + \sum_{j=1}^{n} b_{ij}x_j(t-h) + b_i\xi \qquad (i = 1, \ldots, n) \tag{1.1}$$

where $x_i(t) = z_i(t) - z_i^\circ(t)$ are the deviations (disturbances) of the coordinates of the regulated vector quantity $z(t)$ at the output of the regulated object A for the given (undisturbed) motion $z^\circ(t)$; ξ is a scalar quantity, the regulating action of the control B, which is formed on the basis of information on the discrepancy x; h is the time lag ($h = \text{const} > 0$); a_{ij}, b_{ij}, b_i are constants, the parameters of the system.

A characteristic feature of the system is the presence of the time lag (hysteresis) in the regulated object. It is known [7] that for the determination (with $t > t_0$) of some disturbed motion $x(t)$ of a system described by equations with time lag $h > 0$, it is necessary to know the past history of this motion. The solutions $x_i(t)$ $(i = 1, \ldots, n)$ of the equations with time lag are determined by the initial conditions

$$x^\circ(t_0 + \tau) = \{x_i^\circ(t_0 + \tau)\} \qquad (i = 1, \ldots, n; \ -h \leqslant \tau \leqslant 0)$$

It is for this reason that in the sequel we shall mean by initial disturbances $x^\circ(\cdot)$ (for some $t = t_0$) such initial conditions $x^\circ(\cdot) = \{x_i^\circ(t_0 + r)\}$. In this article we restrict ourselves to the stationary case when the coefficients of the equations do not depend on time. The initial instant of time t_0 will, therefore, not play any special role. We shall omit the symbol t_0 whenever this cannot lead to a misunderstanding.

The quantities which describe the state of the system with after effect (hysteresis) at a given time ($t = t_0$), and completely determine its behavior in the future ($t > t_0$) are thus the segments of the trajectory $\{x_i(t_0 + r)\}$ $(-h \leqslant r \leqslant 0)$. But in such a case it must be possible to formulate the control at the given instant t on the basis of the information on the *entire* curve $\{x_i(t + r)\}$ for the preceding time interval $-h \leqslant r \leqslant 0$. In other words, the quantity ξ in the Equations (1.1) must be treated as a functional $\xi(t) = \xi[x(t + r)]$ defined on the curves $x(t + r) = \{x_i(t + r)\}$ $(-h \leqslant r \leqslant 0)$.

Under the conditions that ξ is some functional, the Equations (1.1) become equations of a more general nature than the ordinary equations with time lag. These are equations of the form

$$\frac{dx_i}{dt} = X_i[x(t + \tau)] \qquad (i = 1, \ldots, n) \qquad (X_i - \text{functional}) \tag{1.2}$$

Such equations were considered by various authors. Here, we call attention to the works of Tikhonov [8] and Myshkis [9]. The theory of stability for such equations by the method of Liapunov functions was developed in [3]. For the element of the trajectory of Equation (1.2) corresponding to the instant t, it is convenient to take, here also, the segment $x(t + r)$ (see [3, p. 157]) of this trajectory with $-h \leqslant r \leqslant 0$, and to study the motion of the system in the corresponding function space $\{x_i(r)\}$ $(i = 1, \ldots, n; -h \leqslant r \leqslant 0)$. Unless stated otherwise, we shall take for the space $\{x_i(r)\}$ the space $C_{-h,0}$ of continuous functions $\{x_i(r)\}$ $(i = 1, \ldots, n; -h \leqslant r \leqslant 0)$. We shall denote the elements of $C_{-h,0}$ by $x_c(\cdot)$. The norm will be defined as

$$\|x_c(\cdot)\| = \sup_\tau \Big[\sum_{i=1}^n x_i^2(\tau)\Big]^{1/2} \qquad (-h \leqslant \tau \leqslant 0)$$

2. Statement of the problem. The problem consists in the construction of the control $\xi[x_c(\cdot)]$, which will insure stable operation of the system and minimize the given criterion for the quality of the transfer process*. As a criterion of this type we select the integral of the square error

$$I_\xi[x_c^\circ(\cdot)] = \int_0^\infty \Big[\sum_{i=1}^n x_i^2(x_c^\circ(\cdot), t) + \xi^2[x_c(x_c^\circ(\cdot), t, \cdot)]\Big] dt \qquad (2.1)$$

Under the chosen law for the control $\xi = \xi[x_c(\cdot)]$, the quantity I_ξ is a functional of the initial disturbance $x_c^\circ(\cdot)$. We will call this quantity the index of completeness of the system. The set of admissible controls ξ, from which one should select the optimal control ξ°, is the set Ξ of all functionals $\xi[x_c(\cdot)]$ (not necessarily linear ones) defined on the continuous curves $x_c(\cdot)$, which are elements of the above defined space $C_{-h,0}$ (Section 1), and satisfy the Cauchy-Lipschitz condition

$$\|\xi[x_c^*(\cdot)] - \xi[x_c(\cdot)]\| \leqslant L_\xi \|x_c^*(\cdot) - x_c(\cdot)\| \qquad (2.2)$$

The problem is the following: for known parameters a_{ij}, b_{ij}, and b_i, and for the time lag h of the system (1.1), one is required to specify a law of control $\xi^\circ = \xi^\circ[x_c(\cdot)]$ for which the following conditions are satisfied.

1) The undisturbed motion $x = 0$ is asymptotically stable relative to arbitrary disturbances $x_c^\circ(\cdot)$ in view of Equations (1.1) when $\xi = \xi^\circ[x_c(\cdot)]$.

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* The motion, which corresponds to the initial condition $x_c^\circ(\cdot)$ (when $t_0 = 0$) will be denoted by the symbol $x(x_c^\circ(\cdot), t)$ when we speak of a point $\{x_i(t)\}$, and by the symbol $x_c(x_c^\circ(\cdot), t, \cdot)$ when we speak of a segment of the curve $\{x_i(t + r)\}$ $(-h \leqslant r \leqslant 0)$.

2) For an arbitrary initial condition $x_c^\circ(\cdot)$, the index of the system (2.1), with $\xi = \xi^\circ$, is a minimum for the class of admissible controls Ξ, i.e.

$$I_{\xi^\circ}[x_c^\circ(\cdot)] = \min I_\xi\ [x_c^\circ(\cdot)] \qquad (\xi \in \Xi) \tag{2.3}$$

Note 2.1. Stability is here understood in the sense of its definition as found in [3], i.e. the motion $x = 0$ is stable (in the large) if for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\sum_{i=1}^{n} x_i^2(x_c^\circ(\cdot), t) < \varepsilon^2 \qquad \text{when } t > 0 \tag{2.4}$$

whenever

$$\|x_c^\circ(\cdot)\| \leqslant \delta \tag{2.5}$$

and, furthermore, for every $x_c(\cdot)$ the following limit relation holds

$$\lim \left[\sum_{i=1}^{n} x_i^2(x_c^\circ(\cdot), t)\right] = 0 \qquad \text{for } t \to \infty \tag{2.6}$$

2.2. In place of the system (1.1) one can consider equations of a more general form

$$\frac{dx_i}{dt} = X_i(x_c(t, \cdot), \xi) \tag{2.7}$$

where the X_i are functionals defined on the curves $x_c(\cdot)$ and satisfy the Cauchy-Lipschitz condition in $x_c(\cdot)$ and ξ; ξ is an r-dimensional vector. In this case one can formulate the following problem on the minimum of the criterion of quality, which is more general than (2.1), in the form

$$J_\xi[x_c^\circ(\cdot)] = \int_0^\infty \omega\,[x(x_c^\circ(\cdot), t), \xi\,[x_c(x_c^\circ(\cdot), t, \cdot)]]\,dt \tag{2.8}$$

where $\omega[x, \xi]$ is some positive definite function of its arguments. Such a more general problem is, however, difficult to solve.

We note, however, that all the following arguments remain valid (when I_ξ is given by (2.1)) in any case for equations of the form

$$\frac{dx_i}{dt} = \int_{-h}^{0} \sum_{j=1}^{n} x_j(\tau)\,d\eta_{ij}(\tau) + b_i \xi \tag{2.9}$$

where on the right-hand side we have a Stieltjes integral. We shall confine our consideration to the case (1.1) in order not to complicate the presentation.

An analogous problem without time lag was considered by Letov [6] who derived the equations for the optimal control ξ° by the methods of dynamic programming. The question on the solvability of these equations, under the condition of stability of the system, was studied in [10]. In that paper there were given conditions for the stability of a linear system with a single control ξ. An analogous problem, for the case when the equations of the disturbed motion contain random parameters, was investigated in [11,12].

The considered problems fall into the class of problems on optimal control. The statement and original investigation of such problems were made by Fel'dbaum [13] (see also [14], where there are given basic results and a bibliography). The development of the mathematical theory for the case of ordinary equations was accomplished by Pontriagin and his school [15]. Problems of optimal control in systems with time lag (and in more general systems with distributed parameters) were considered by several authors. We mention the works of Kramer [16] and Bellman and Kalabe [17], which are similar in subject matter to our work. We note also that some problems were treated by Butkovskii and Lerner [18] and Kharatishvili [19], but in a way different from ours.

The aim of the present article is the description of a solution of the above stated problem for the system (1.1) on the basis of those considerations which led to the development of the method of Liapunov functions for systems with hysteresis. We call attention to the fact that the presented method of solution makes it possible to give the optimal control ξ° in explicit form.

3. General approach to the solution of the problem. The following arguments are extensions of the considerations of [10-12] to the case of equations with time lag. Hereby, the functions of Liapunov are essentially replaced by corresponding functionals.

We will formulate and prove sufficient conditions for the optimum of the control ξ°. We consider the problem in its general formulation given in the Note 2.2. One should, however, realize that the construction of the functionals appearing in Theorem 3.1 is quite difficult in the general case. In what follows we confine ourselves to the application of Theorem 3.1 only to the problem stated in Section 2 for the system (1.1).

Theorem 3.1. If it is possible to give a functional $v[\, x_c(\cdot)\,]$ which is positive definite, which has an arbitrarily small upper limit* which satisfies the condition

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* We make use here of the terminology of [3, pp.150-170] (in the metric $C_{-h,0}$).

$$\lim v[x_c(\cdot)] = \infty \qquad \text{for } \|x_c(\cdot)\| \to \infty$$

and is such that the derivative $(dv/dt)_\xi$, in view of (2.7), satisfies the conditions

$$\left(\frac{dv}{dt}\right)_{\xi^\circ} + \omega[x(0), \quad \xi^\circ[x_c(\cdot)]] = 0 \qquad (x_c(\cdot) \in C_{-h,0}) \tag{3.1}$$

for some functional $\xi^\circ \in \Xi$, and

$$\left(\frac{dv}{dt}\right)_{\xi^\circ} + \omega[x_c(0), \quad \xi^\circ[x_c(\cdot)]] = \min_{\xi \in \Xi}\left(\left(\frac{dv}{dt}\right)_\xi + \omega(x(0), \quad \xi[x_c(\cdot)]]\right)$$
$$(x_c(\cdot) \in C_{-h,0}) \tag{3.2}$$

then the next inequality holds

$$I_{\xi^\circ}[x_c(\cdot)] = v[x_c(\cdot)] \tag{3.3}$$

$$I_{\xi^\circ} = \min_{\xi \in \Xi} I_\xi \tag{3.4}$$

and, hence, the functional $\xi^\circ[x_c(\cdot)]$, satisfying the conditions (3.1) and (3.2), will be an optimal control.

Note 3.1. If one considers the problem with an additional restriction (for example $\|\xi\| = \sqrt{\xi_1^2 + \ldots + \xi_r^2} < 1$), then the minimum (3.1) must be determined under auxiliary restrictions. It is sufficient that the functional v be positive definite only on the curves $x_c(\cdot)$ satisfying the Lipschitz conditions [3].

Proof of the theorem. The asymptotic stability in the large of the solution $x = 0$, under the conditions of Theorem 3.1, follows from the theorems given in [3] because the functional $v[x_c(\cdot)]$ has a negative definite derivative $(dv/dt)_{\xi^\circ} = -\omega[x, \xi^\circ]$. Thus, the condition 1 (Section 2) is satisfied.

Let us verify the fulfillment of condition 2. Integration of (3.1) from $t = 0$ to $t = \infty$ (which is valid due to asymptotic stability) leads to

$$v[x_c^\circ(\cdot)] = \int_0^\infty \omega[x(x_c^\circ(\cdot), t)_{\xi^\circ}, \xi^\circ[x_c(x_c^\circ(\cdot), t, \cdot)_{\xi^\circ}]]\, dt = I_{\xi^\circ}[x_c^\circ(\cdot)] \tag{3.5}$$

where the subscript ξ° of x indicates that this is the solution of the equations (2.7) when $\xi = \xi^\circ$. The equality (3.5) establishes the Equation (3.3).

Next we consider some admissible control satisfying condition 1. In consequence of (3.2) we have

$$\left(\frac{dv}{dt}\right)_{(\xi)} \geqslant -\omega\,[x(0), \xi\,[x_c(\cdot)]] \tag{3.6}$$

Integrating this inequality from $t = 0$ to $t = \infty$, we obtain

$$v\,[x_c^\circ(\cdot)] \leqslant \int_0^\infty \omega\,[x\,(x_c^\circ(\cdot), t)_\xi, \xi\,[x_c\,(x_c^\circ(\cdot), t(\cdot)_\xi]]\,dt = I_\xi\,[x_c^\circ(\cdot)] \tag{3.7}$$

from which the validity of (3.4) follows in view of (3.5). This establishes the theorem.

4. Equations for the optimal control ξ°. The basic results of this article are contained in the next theorem.*

Theorem 4.1. If the system can be stabilized** with some admissible control $\xi \in \Xi$, then there exists an optimal control $\xi^\circ[x_c(\cdot)]$ which is a linear functional on the curve $x_c(\cdot) = \{x_i(r)\}$ $(i = 1, \ldots, n;\ -h \leqslant r \leqslant 0)$ which has the form

$$\xi^\circ\,[x_c(\cdot)] = \sum_{i=1}^n \alpha_i x_i(0) + \int_{-h}^0 \left\{\sum_{i=1}^n \beta_i(\tau)\, x_i(\tau)\right\} d\tau \qquad (\alpha_i = \text{const}) \tag{4.1}$$

The optimal index of completeness of the system I_{ξ° is a quadratic functional on the curves $x_c(\cdot)$. This functional has the form

$$I_{\xi^\circ}\,[x_c(\cdot)] = \sum_{i,j=1}^n \alpha_{ij} x_i(0)\, x_j(0) + \int_{-h}^0 \left\{\sum_{i,j=1}^n \beta_{ij}(\tau)\, x_i(0)\, x_j(\tau)\right\} d\tau +$$

$$+ \int_{-h}^0 \int_{-h}^0 \left\{\sum_{i,j=1}^n \gamma_{ij}(\tau, \vartheta)\, x_i(\tau)\, x_j(\vartheta)\right\} d\tau d\vartheta = v\,[x_c(\cdot)] \tag{4.2}$$

$$(\alpha_{ij} = \text{const}, \quad \gamma_{ij}(\tau, \vartheta) = \gamma_{ji}(\vartheta, \tau))$$

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* We restrict ourselves to the case when the matrix $\|b_{ij}\|_1^n$ in Equations (1.1) is nonsingular. In case the matrix is singular, the arguments are carried through in the same way, except that in the definition of the positive definiteness one has to introduce certain changes (see below, Note 5.1, Section 5).

** In other words, if one can give a control $\xi \in \Xi$ for which the solution $x = 0$ of the system (1.1) is asymptotically stable and the integral I_ξ converges.

Note 4.1. Under the hypotheses of the theorem it is assumed that it is possible to stabilize the system by means of at least one admissible control ξ. The answer to the question on this possibility is a separate problem. We note that the process of the construction of a solution described below, in Section 6, leads also to the solution of the problem on the possibility of stabilization (see Section 6). One can also give, in closed form, certain sufficient conditions for the possibility of stabilization. This problem will be treated in a separate paper. Here we only note that the stabilization is certainly possible if the system (1.1) is asymptotically stable when $\xi = 0$, or if the time lag h is sufficiently small (or if the numbers b_{ij} are small), and if for the system

$$dx_i/dt = \sum a_{ij}x_j + b_i\xi$$

the stabilization conditions given in the article [10] are satisfied.

The conditions are as follows. It is necessary and sufficient that the root space of the matrix $A = \| a_{ij} \|_1^n$, corresponding to the roots λ with nonnegative real parts, lies in the space formed by the vectors b, Ab, ..., $A^{n-1}b$.

The proof of the theorem will be indicated in Section 6. Assuming that the theorem is true, we derive now the equations which are satisfied by the quantities a_{ij}, β_{ij} and γ_{ij} which determine the optimal index of the system I_{ξ° and the optimal control ξ°.

These equations are obtained by substituting v from (4.2) and the quantity $\omega = x_1^2 + \ldots + x_n^2 + \xi^2$ into the conditions (3.1) and (3.2). Let us first construct the expression for the derivative $(dv/dt)_\xi$. For the computation of this derivative one must, in accordance with the general rule ([3, pp.170-179]), substitute in (4.2) for $x(r)$ and $x(\vartheta)$ the running segment of the trajectory $x(t + r)$ and $x(t + \vartheta)$ and differentiate with respect to t. The change of the functions $x(t + r)$ and $x(t + \vartheta)$ is a shift to the right. One can assume that these functions are differentiable with respect to t, r and ϑ (see [3, pp.158,162]). The indicated differentiation reduces to the transformation of the functions $x(t + r)$, $x(t + \vartheta)$ by means of an operator ([3, pp.160-166]).

$$y(t+\tau) = x_t'(t+\tau) = \begin{cases} x_\tau'(t+\tau) & (-h \leqslant \tau < 0) \\ \sum_{j=1}^{n} a_{ij}x_j(t) + \sum_{j=1}^{n} b_{ij}x_j(t-h) + b_i\xi & (\tau = 0) \end{cases} \tag{4.3}$$

$$y(t+\vartheta) = x_t'(t+\vartheta) = \begin{cases} x_\vartheta'(t+\vartheta) & (-h \leqslant \vartheta < 0) \\ \sum_{j=1}^{n} a_{ij}x_j(t) + \sum_{j=1}^{n} b_{ij}x_j(t-h) + b_i\xi & (\vartheta = 0) \end{cases}$$

Taking this into consideration, and transforming the expression obtained from the differentiation of the functional (4.2) through integration by parts,* we obtain the next equation which gives the derivative (when $t = 0$):

$$\left(\frac{dv}{dt}\right)_\xi = \sum_{i,j,k} (\alpha_{ij}a_{jk} + \alpha_{kj}a_{ji})\, x_i(0)\, x_k(0) +$$

$$+ \sum_{i,j,k} (\alpha_{ij}b_{jk}x_i(0)\, x_k(-h) + \alpha_{ij}b_{ik}x_j(0)\, x_k(-h)) +$$

$$+ \sum_{i,j} (\alpha_{ij}b_j x_i(0) + \alpha_{ij}b_i x_j(0))\, \xi + \sum_{i,j,k} [a_{ik}x_k(0) +$$

$$+ b_{ik}x_k(-h)] \int_{-h}^{0} \beta_{ij}(\tau)\, x_j(\tau)\, d\tau + \sum_{i,j} \xi b_i \int_{-h}^{0} \beta_{ij}(\tau)\, x_j(\tau)\, d\tau +$$

$$+ \sum_{i,j} \beta_{ij}(0)\, x_i(0)\, x_j(0) - \sum_{i,j} \beta_{ij}(-h)\, x_i(0)\, x_j(-h) -$$

$$- \int_{-h}^{0} \Big\{ \sum_{i,j} \beta_{ij}'(\tau)\, x_j(\tau)\, x_i(0) \Big\} d\tau + \qquad (4.4)$$

$$+ 2 \int_{-h}^{0} \Big\{ \sum_{i,j} [\gamma_{ij}(0, \vartheta)\, x_i(0)\, x_j(\vartheta) - \gamma_{ij}(-h, \vartheta)\, x_i(-h)\, x_j(\vartheta)] \Big\} d\vartheta -$$

$$- \int_{-h}^{0} \int_{-h}^{0} \Big[\sum_{i,j} \Big[\frac{\partial \gamma_{ij}(\tau, \vartheta)}{\partial \tau} + \frac{\partial \gamma_{ij}(\tau, \vartheta)}{\partial \vartheta} \Big] x_i(\tau)\, x_j(\vartheta) \Big] d\tau\, d\vartheta$$

We add $x_1^2 + \ldots + x_n^2 + \xi^2$ to the right hand side and equate the obtained sum to zero. In view of (3.1), we thus obtain the first equation for the quantities v and ξ°. The second equation is obtained by differentiating the first one with respect to ξ, because the left hand side of this equation will have a minimum, when $\xi = \xi^\circ$, in view of (3.2).

Solving this second equation for ξ°, we obtain

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* The validity of this operation follows from the fact that the functions $\beta_{ij}(\tau)$ and $\gamma_{ij}(\tau, \vartheta)$ are sufficiently smooth (see Section 6).

$$\xi^\circ = -\left\{\sum_{i,j} \alpha_{ij} b_j x_i(0) + \frac{1}{2}\int_{-h}^{0} \sum_{i,j} b_i \beta_{ij}(\tau)\, x_j(\tau)\, d\tau\right\} \tag{4.5}$$

Thus, the problem reduces to the determination of functionals v and ξ° satisfying the conditions (4.5) and

$$\left(\frac{dv}{dt}\right)_{\xi^\circ} + \sum_{i=1}^{n} x_i^2(0) + \xi^2 = 0 \tag{4.6}$$

where $(dv/dt)_\xi$ has the form (4.4).

Substituting (4.4) and (4.5) into the Equation (4.6), and equating the coefficients of $x_i(\vartheta)$ and $x_i(r)$ to zero, we obtain the system of equations for the functions $\beta_{ij}(r)$, $\gamma_{ij}(r, \vartheta)$, and for the constants a_{ij}. The resulting equations are quite complicated, but it is possible to obtain their approximate numerical solutions. One can, for example, solve these equations by expanding the functions $\beta_{ij}(r)$ and $\gamma_{ij}(r, \vartheta)$ into Fourier series. This approach is especially convenient and justified, because it is sufficient to approximate the functions β_{ij}, and γ_{ij} (which determine the optimal control ξ° (4.5)) in the mean in order to find an approximate optimal system. The technical difficulty of the solution of the equations for a_{ij}, β_{ij} and γ_{ij} is also due to the fact that one has the find those solutions for which the functional (3.2) is positive definite. The difficulty is circumvented by the approximate method of solution described in Section 6, where the problem is reduced to the successive solution of a system of linear equations. Hereby one obtains asymptotically those solutions for which the requirements of positive definiteness of the functional (4.2) is satisfied.

It thus follows that an optimal control ξ°, in a system with time lag (1.1) under the conditions that the quantity (2.1) be a minimum, is an ideal control [20, p. 360] which feeds into the input of the regulated object A at any instant t the quantity $\xi^\circ[x(t, \cdot)]$ (4.1) that is produced on the basis of the measurements of the discrepancy in x at the given instant of time, and during the preceding instants $t - h < r < t$. The results of the measurements of the preceding values of the discrepancies are hereby transformed in the integrating links that compute the integrals

$$\int_{-h}^{0} \beta_i(\tau)\, x_i(t+\tau)\, d\tau$$

It is interesting to note that for the system (1.1) with a discrete time lag h, the optimum control ξ° is transformed into a form which contains integrals that account for the continuous effect of the hysteresis during the entire interval of the time lag.

5. Auxiliary material from the theory of Liapunov's second method. In this section we present some information of an auxiliary nature. These results are used below in Section 6 for the construction of the functional v (4.2) which solves the optimization problem.

Let us consider the linear equation with hysteresis

$$\frac{dx_i}{dt} = \sum_{j=1}^{n} c_{ij}x_j(t) + \sum_{j=1}^{n} b_{ij}x_j(t-h) + \int_{-h}^{0} \sum_{j=1}^{n} d_{ij}(\tau)\, x_j(\tau)\, d\tau \qquad (5.1)$$

$$(c_{ij} = \text{const}, \quad b_{ij} = \text{const})$$

where the $d_{ij}(\tau)$ are differentiable functions of the variable τ. The matrix $\|b_{ij}\|_1^n$ is assumed to be nonsingular (see footnote in Section 4).

Let us consider the problem on the construction of the functional V for the Equation (5.1). This functional plays the role of a Liapunov function and has a given derivative dV/dt in view of the equations (5.1).

For ordinary linear differential equations it is most convenient to use Liapunov functions that are quadratic forms. It is natural that for the equations with hysteresis (5.1) quadratic functionals $V[x_c(\cdot)]$ must play a similar role.

In this article we restrict ourseves to the case when the derivative dV/dt has (for $t = 0$) the form

$$\frac{dV}{dt} = \sum_{i,j=1}^{n} \omega_{ij}x_i(0)\, x_j(0) + \int_{-h}^{0} \Big\{ \sum_{i,j=1}^{n} \nu_{ij}(\tau)\, x_i(0)\, x_j(\tau)\Big\} d\tau +$$

$$+ \sum_{i,j=1}^{n} \lambda_{ij}x_i(0)\, x_j(-h) + \int_{-h}^{0} \Big\{ \sum_{i,j=1}^{n} \varphi_{ij}(\tau)\, x_i(-h)\, x_j(\tau)\Big\} d\tau + \qquad (5.2)$$

$$+ \int_{-h}^{0}\int_{-h}^{0} \Big\{ \sum_{i,j=1}^{n} \varepsilon_{ij}(\tau,\vartheta)\, x_i(\tau)\, x_j(\vartheta)\Big\} d\tau d\vartheta = F[x_c(\cdot)] \qquad ((\nu_{ij}, \varphi_{ij}, \varepsilon_{ij}) \in C_1)$$

The theory can, however, be developed without difficulty also for more general cases, when $F[x_c(\cdot)]$ is a quadratic functional of $x_c(\cdot)$ of a more general nature,* and also for the case when the right-hand sides

* The author considers it to be his duty to note that, independently of him, the theory of the construction of the functionals v that have a quadratic derivative, for linear equations with time lag, was developed recently by Iu.M. Repin. The results of Repin have been presented in a report "On some functionals for equations with time lag" at the 4th All-Union Mathematical Congress (Program of the Congress, p. 75).

of the Equations (5.2) depend on time t. The needed material for such a development in analogy with the known theory with ordinary differential equations, can be found, for example, in [3].

The following theorem is valid.

Theorem 5.1. Suppose that the solutions of the Equations (5.1) are asymptotically stable, i.e. the roots of the equations

$$\det \| J_{kl} - \delta_{kl}\lambda \| = 0$$

$$J_{kl} = c_{kl} + b_{kl}e^{-\lambda h} + \int_{-h}^{0} d_{kl}(\tau)\, e^{\lambda\tau}\, d\tau \qquad \begin{pmatrix} \delta_{kl} = 0 & \text{for } k \neq l \\ \delta_{kl} = 1 & \text{for } k = l \end{pmatrix}$$

have negative real parts [3, p. 164]. Then for every functional of the form (5.2) there exists one quadratic functional V which has the form

$$V[x_c(\cdot)] = \sum_{i,j=1}^{n} \alpha_{ij} x_i(0) x_j(0) + \sum_{i,j=1}^{n} \left[\int_{-h}^{0} \{\beta_{ij}(\tau)\, x_j(\tau)\, x_i(0)\}\, d\tau \right] + \int_{-h}^{0}\int_{-h}^{0} \left\{ \sum_{i,j=1}^{n} \gamma_{ij}(\tau, \vartheta)\, x_i(\tau)\, x_j(\vartheta) \right\} d\tau d\vartheta$$

$$(\beta_{ij},\ \gamma_{ij} \text{ - are piece-wise smooth functions}) \tag{5.4}$$

whose derivative satisfies the Equation (5.2) in view of the Equations (5.1).

Proof. In [3] it is shown that a functional $V[x_c(\cdot)]$, which has a derivative dV/dt and satisfies the condition (5.2), is of the form

$$V[x_c(\cdot)] = -\int_{0}^{\infty} F[x_c(x_c(\cdot), t, \cdot)]\, dt \tag{5.5}$$

For the proof of the theorem it is, therefore, sufficient to show that the functional (5.5) actually has the form (5.4). The verification of this can be accomplished by starting out with general theorems of functional analysis, or directly with the construction of the functional first on piece-wise constant initial functions $x_i(t)$ and then going over to continuous functions $x_i(r)$. This will not lead to any great difficulties because the solutions $\{x_i(\cdot),\ t\}$, where $x(\cdot)$ is a function of the form

$$x_i(\tau) = 0 \quad (i = 1, \ldots, n;\ i \neq j,\ -h \leqslant \tau \leqslant 0)$$

$$x_j(\tau) = 0 \quad (-h \leqslant \tau < \tau_1,\ \tau_2 < \tau \leqslant 0), \qquad x_j(\tau) = \zeta \quad (\tau_1 < \tau < \tau_2)$$

or

$$x_j(\tau) = 0 \qquad (-h \leqslant \tau < 0), \qquad x_j(0) = \zeta$$

are continuously differentiable in r_1, r_2 and ζ.

The verification of the smoothness of the functions β_{ij} and γ_{ij} thus constructed, can be carried out on the basis of the property of the integrability and differentiability of the solutions (5.1) with respect to the initial curves. We omit the technical details.

Note 5.1. One can show that on the initial curves $x_c(\cdot)$ that satisfy the Cauchy-Lipschitz conditions

$$|x_i(\tau_1) - x_i(\tau_2)| \leqslant L|\tau_2 - \tau_1| \quad \text{for} \quad \|x_c(\cdot)\| \leqslant 1$$
$$|x_i(\tau_1) - x_i(\tau_2)| \leqslant L\|x_c(\cdot)\| |\tau_2 - \tau_1| \quad \text{for} \quad \|x_c(\cdot)\| > 1$$
$$(L = \text{const}, \ i = 1, \ldots, n)$$

the functional V is positive definite if the functional F is negative definite on these curves.* This is sufficient for what follows, because, in view of the statements made in [3, p. 158], we can restrict ourselves to such curves.

6. Construction of the optimal control ξ° by deforming the system. In this section we describe the process of the construction of the optimal control ξ°, and of the functional v by the method of the deformation of the system through the introduction of a parameter μ. The arguments given here are similar to those found in [10-12] for analogous cases.

Let us consider an auxiliary problem. Suppose we are to find the optimal vector control

$$\xi[x_c(\cdot), \mu] = \{\xi_1[x_c(\cdot), \mu], \ldots, \xi_n[x_c(\cdot), \mu]\}$$

which minimizes the quantity

$$J[x_c^\circ(\cdot)]_\xi^\mu = \int_0^\infty \left\{\sum_{i=1}^n x_i^2(t) + \sum_{i=2}^n (1-\mu)\,\xi_i^2[x_c(t,\cdot), \mu] + \xi_1^2[x_c(t,\cdot), \mu]\right\} dt \qquad (6.1)$$

---

* Under the condition that the matrix $\|b_{ij}\|_1^n$ is nonsingular. One can free himself of this restriction by introducing in a special way a variation of the concept of positive definiteness, namely, by requiring positive definiteness of v with respect to $\|x(0)\|$ only on the curves

$$\|x_c(\cdot)\| \leqslant \|x(0)\|$$

in a control system described by the equations

$$\frac{dx_i}{dt} = \mu \left[\sum_{j=1}^{n} a_{ij}x_j(t) + \sum_{j=1}^{n} b_{ij}x_j(t-h) + b_i\xi_1\right] + (1-\dot{\mu})\,\xi_i$$

$$(i = 1, \ldots, n) \tag{6.2}$$

When $\mu = 0$, the problem can be solved by elementary means. Indeed, when $\mu = 0$ the system has no time lag and the functional v reduces simply to the function

$$v(x_1, \ldots, x_n) = v[x_c(\cdot), \mu = 0].$$

For such a function* we derive, with the aid of (3.1) and (3.2), the equations

$$\sum_{i=1}^{n} \frac{\partial v}{\partial x_i}\xi_i + \sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} \xi_i^2 = 0, \qquad 2\xi_i + \frac{\partial v}{\partial x_i} = 0 \qquad (i = 1, \ldots, n)$$

These equations are satisfied by

$$v = \sum_{i=1}^{n} \alpha_{ii}x_i^2 \qquad \left(\alpha_{ii} = \frac{1}{\sqrt{2}}\right)$$

The optimal control has the form

$$\xi_i[x_c(\cdot)] = -x_i(0) \qquad (i = 1, \ldots, n)$$

Let us assume now that the problem has been solved for some value of μ. This means that we have found functionals $v[x_c(\cdot), \mu]$ and $\xi_i[x_c(\cdot), \mu]$ which have the form

$$v[x_c(\cdot), \mu] = \sum_{i,j} \alpha_{ij}(\mu)\,x_i(0)\,x_j(0) + \int_{-h}^{0} \left\{\sum_{i,j} \beta_{ij}(\tau, \mu)\,x_i(0)\,x_j(\tau)\right\} d\tau + $$
$$+ \int_{-h}^{0}\int_{-h}^{0} \left\{\sum_{i,j} \gamma_{ij}(\tau, \vartheta, \mu)\,x_i(\tau)\,x_j(\vartheta)\right\} d\tau d\vartheta \tag{6.3}$$

$$\xi_1[x_c(\cdot), \mu] = -\left\{\sum_{i,j} [\alpha_{ij}(\mu)\,b_i\mu + \alpha_{1j}(1-\mu)]\,x_j(0) +, \right.$$
$$\left. + \frac{1}{2}\int_{-h}^{0} \left\{\sum_{i,j} [\beta_{ij}(\tau, \mu)\,b_i\mu + \beta_{1j}(\tau, \mu)(1-\mu)]\,x_j(\tau)\right\} d\tau\right\} \tag{6.4}$$

---

* One inconvenience that arises here is the fact that $v(x_1, \ldots, x_n)$ is positive indefinite in $\{x_c(\cdot)\}$, but this is not important at this point (see the preceding footnote).

$$\xi_i [x_c(\cdot), \mu] = -\left\{\sum_j \alpha_{ij}(\mu)\, x_j(0) + \frac{1}{2}\int_{-h}^{0} \left\{\sum_j \beta_{ij}(\tau, \mu)\, x_j(\tau)\right\} d\tau\right\}$$
$$(i = 2, \ldots, n)$$

and satisfy the Equations (3.1) and (3.2), i.e. have found a functional v satisfying the equations

$$\sum_{i,j,k} (\alpha_{ij}(\mu)\, a_{jk} + \alpha_{kj}(\mu)\, a_{ji})\, \mu x_i(0)\, x_k(0) +$$
$$+ 2\sum_{i,j,k} \mu\alpha_{ij}(\mu)\, b_{jk} x_i(0)\, x_k(-h) + \sum_{i,j,k} [a_{ik} x_k(0) + b_{ik} x_k(-h)]\, \mu \times$$
$$\times \int_{-h}^{0} \beta_{ij}(\tau, \mu)\, x_j(\tau)\, d\tau - \left\{\sum_{i,j} [\alpha_{ij}(\mu)\, b_i\mu + \alpha_{1j}(1-\mu)]\, x_j(0) + \right. \quad (6.5)$$
$$\left. + \frac{1}{2}\int_{-h}^{0} \left\{\sum_{i,j} [\beta_{ij}(\tau, \mu)\, b_i\mu + \beta_{1j}(\tau, \mu)(1-\mu)]\, x_j(\tau)\right\} d\tau\right\}^2 -$$
$$- \sum_{i=2}^{n} \left\{\sum_j \alpha_{ij}(\mu)\, x_j(0) + \frac{1}{2}\int_{-h}^{0} \left\{\sum_j \beta_{ij}(\tau, \mu)\, x_j(\tau)\right\} d\tau\right\}^2 +$$
$$+ \int_{-h}^{0}\int_{-h}^{0} \left\{\sum_{i,j} \gamma_{ij}(\tau, \vartheta, \mu)\, x_i(\tau)\, x_j'(\vartheta) + \right.$$
$$\left. + \gamma_{ij}(\tau, \vartheta, \mu)\, x_i'(\tau)\, x(\vartheta)]\right\} d\tau d\vartheta + \sum_i x_i^2(0) = 0$$

Let us consider the question on how the functionals $v[x_c(\cdot), \mu]$ and $\xi_i[x_c(\cdot), \mu]$ change with a change of the parameter $(0 \leqslant \mu \leqslant 1)$. For this purpose we differentiate formally the Equations (6.5) with respect to μ. After some simple transformations we obtain

$$\left(\frac{dV}{dt}\right)_{(6.2),\,\mu} = -\sum_{i,j,k} (\alpha_{ij}(\mu)\, a_{jk} + \alpha_{kj}(\mu)\, a_{jk})\, x_i(0)\, x_j(0) -$$
$$- 2\sum_{i,j,k} \alpha_{ij}(\mu)\, b_{jk} x_i(0)\, x_k(-h) -$$
$$\sum_{i,j,k} [a_{ik} x_k(0) + b_{ik} x_k(-h)] \int_{-h}^{0} \beta_{ij}(\tau, \mu)\, x_j(\tau)\, d\tau \quad (6.6)$$

Here $V = \partial v/\partial\mu$, and the symbol (dV/dt) denotes the derivative of the functional V in view of the Equation (6.2) for the chosen value μ, and for the values ξ_i equal to the functionals (6.4) that define the optimal control for the given μ. But because of Theorem 5.1, the functional V, satisfying the conditions (6.6), exists and has the form (5.4) (because for the given μ the system (6.2) is asymptotically stable) and, hence, the performance of the differentiation was valid. From the expression of $\partial v/\partial\mu$ we find also $\partial\xi/\partial\mu$ by means of the Formulas (6.4). Making use of the fact that the derivative $\partial v/\partial\mu$ exists, one can show that the solution of the problem found for $\mu = 0$ can be extended over the entire interval $0 \leqslant \mu \leqslant 1$ (if the system can be stabilized for $\mu = 1$) and one thus obtains the solution of the problem when $\mu = 1$, which coincides with the problem originally set. This establishes the Theorem 4.1. Since the arguments here are the same as those in [11], they will be omitted.

In conclusion, we note that the method described above shows the procedure for solving the problem approximately: we solve the problem when $\mu = 0$, and then find the increments of the functions $\Delta v[x_c(\cdot), \mu]$ and $\Delta\xi_i[x_c(\cdot), \mu]$ that correspond to the increases $\Delta\mu$, by the use of the Equation (6.6) and by setting $\Delta v \approx \Delta\mu\,\partial v/\partial\mu$, $\Delta\xi_i \approx \Delta\mu\,\partial\xi_i/\partial\mu$. The Equations (6.6) that determine $\partial v/\partial\mu$ must be solved approximately here also. If the quantity $\Delta\mu \to 0$, and the solutions of the Equations (6.6) are found with a sufficient degree of accuracy (at least in the mean), then the approximate solution $\xi^{\circ}_{\Delta\mu}$ will converge to the exact solution. We note also that in the case when the system cannot be stabilized with $\mu = 1$, then the functional $v[x_c(\cdot), \mu]$ will be unbounded on some initial curve as $\mu \to 1$. This means that one of the coefficients $a_{ij}(\mu)$, or one of the functions $\beta_{ij}(r, \mu)$ or $\gamma_{ij}(r, \vartheta, \mu)$ become infinitely large.

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